International Review of

Accounting, Banking and Finance

IRABF

Ⓒ2021

Pricing Average Interest Rate Options in the LIBOR Market Model

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A B S T R A C T

This paper employs the LIBOR market model (LMM) to price average interest rate options, which provides an alternative instrument to hedge interest rate risks at a lower cost. The forward LIBOR rates, modeled in the LMM, exhibit positive rates and are market-observable, which avoids pricing errors arising from negative rates and is easier for calibration. The underlying average rate is calculated by summing LIBOR rates rather than integrating instantaneous short rates, which makes our resulting formulas consistent with market practice. The resulting pricing formulas of average interest rate options are shown to be accurate as compared with the Monte Carlo simulation. The calibration procedure and its practical implementation are also examined.

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1. Introduction

Hedging against interest rate risks has become one of the most important tasks for a financial manager. To manage these risks, many interest rate derivatives, such as forward rate agreements, caps, floors, swaps, and swaptions, have been developed and traded actively. These days, besides achieving hedging needs, hedgers further want to reduce their hedging costs, which gives a big challenge and a new direction to financial engineers. To accomplish this purpose, average interest rate options (AIROs) have recently been developed to lower the cost of hedging interest rate risks, and thus become increasingly popular.

Interest rate caps are one of the most actively traded interest rate derivatives. It is well known that a cap is a portfolio of interest rate call options which pay the holder some pre-specified market interest rates minus a cap rate (if positive) or zero (if non-positive), on a pre-agreed notional principal. Therefore, an interest rate cap can be employed to hedge separately interest rate risks of the future cash flows by putting a ceiling on their borrowing interest costs at a cap rate. However, some investors may desire to hedge their average interest costs of overall cash flows rather than individual ones. To achieving this purpose, AIROs are developed and provide for financial practitioners a cheaper and more efficient hedging tool.

AIROs are interest rate options, whose underlying rates are calculated by an arithmetic average of some pre-specified interest rates (e.g. LIBOR rates) over a given time interval. Unlike interest rate caps, AIROs can ceil the average (rather than individual) interest costs at a cap rate. Moreover, if an investor wants to hedge average interest costs, hedging with an AIRO is cheaper than with the corresponding caps (or floors).[1] Since most financial institutions take huge and complicated positions involved with interest rate risks by issuing many kinds of interest rate-related products, such as interest rate-linked structure notes, interest rate swaps, etc., AIROs can be employed to hedge the overall average interest rate risks in a more efficient way. In addition, AIROs are less liable to unanticipated events or the market manipulation by the options' counterparties since their final payoffs depend on the average interest rate during their life, which makes AIROs become a more trustworthy hedging tool. Due to these advantages, AIROs have been widely-traded in the over-the-counter market.

Some earlier research has been conducted on the pricing of AIROs. Within the Vasicek (1977) interest rate model, Longstaff (1995) derives analytic pricing formulas for AIROs. However, the underlying average rate is calculated continuously on the basis of abstract short rates rather than LIBOR rates, which is inconsistent with market practice.[2] Based on the Hull and White (1990, HW) interest rate model, Cheuk and Vorst (1999, CV) modify the setting of the underlying average rate in Longstaff (1995) by computing discretely the arithmetic average of LIBOR rates. However, the LIBOR rates in CV (1999) are transformed from the market-unobservable short rates, and the transformation process would make the resulting pricing formula more complicated. Moreover, the short rates specified in both Vasicek and HW are Gaussian processes, so the negative rates may occur and lead to some pricing error.[3] In addition, the HW model cannot capture the correlations between rates of different terms as realistically as the LIBOR market model (hereafter, LMM), and this may affect the accuracy of pricing AIROs.

The main purpose of this paper is to price AIROs within the LIBOR market model (LMM) framework. The LMM is developed by Brace, Gatarek, and Musiela (1997, BGM), Musiela and

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[1] This statement is proved in Appendix A.

[2] The underlying average rate in Longstaff (1995) is $A(T) = \left(\int_0^T r_u du\right)/T$, where r_t is the instantaneous short rate. However, market practice is that $A(T) = \sum_{i=1}^n L(t_i, t_i)/n$, where $L(t_i, t_i)$ stands for LIBOR rate observed at time t_i.

[3] Rogers (1996) indicated that the Gaussian term structure model has an important theoretical limitation: the rate can attain negative values with positive probability which may cause some pricing error in many cases.

Rutkowski (1997), and Miltersen, Sandermann, and Sondermann (1997). The rate specified in the LMM is a market-observable LIBOR rate which is commonly used in the financial industry. Therefore, pricing AIROs based on the LMM can avoid the complicated transformation from short rates to LIBOR rates, such as Longstaff (1995) and CV (1999), making the resulting pricing formula full of financial-economic intuitions. The resulting pricing formula of AIROS bears resemblance to the Black (1976) pricing model for options on futures in the environment of stochastic interest rates, and thus provides end-users a familiarity to use it. The LIBOR rate modeled in LMM is lognormally distributed, preventing the negative rate problem. The most important advantage of the LMM over the short rate models is its ease and flexibility in the parameter calibration. The LMM can simultaneously calibrate the market-quoted cap volatilities and the correlation matrix of the underlying forward LIBOR rates. Equipped with these advantages, pricing AIROs under the LMM is more suitable for practical implementation.

The paper is organized as follows. Section 2 specifies the approximate lognormal LMM model and introduces different approximate lognormal dynamics under the mechanism of changing the numéraire. Section 3 outlines the contracts of a general AIRO and presents an approximation method to derive the closed-form solutions of the AIRO. Section 4 provides the calibration procedure and examines the accuracy of the approximate formulas based on the Monte Carlo simulation. The conclusion is made in the last section.

2. The Model

We assume that trading takes place continuously in time over an interval $[0,\mathcal{T}]$, $0 < \mathcal{T} < \infty$. The uncertainty is described by the filtered spot martingale probability space $\left(\Omega, \mathcal{F}, \mathcal{Q}, \{\mathcal{F}_t\}_{t \in [0,\tau]}\right)$ where the filtration is generated by the independent standard Brownian motions $Z(t) = (Z_1(t), Z_2(t), \ldots, Z_m(t))$. Note that $\mathcal{Q}$ represents the spot martingale probability measure. We list the notations as follows.

$B(t,T)$= the time-t price of a zero-coupon bond (ZCB) paying one dollar at time T.
$F(t,T)$= the forward LIBOR rate contracted at time t and applied to the period $[T, T + \delta]$ with $0 \leq t \leq T \leq T + \delta \leq \mathcal{T}$.
$\mathcal{Q}^T$= the martingale measure with respective to the numéraire $B(\cdot, T)$.

The relationship between $F(t,T)$ and $B(t,T)$ can be expressed as follows:
$$F(t,T) = \tfrac{1}{\delta}(B(t,T) - B(t,T + \delta))/B(t,T + \delta). \tag{1}$$

Based on the result of Heath, Jarrow, and Morton (1992), BGM (1997) models interest rates in terms of the forward LIBOR rates. We specify briefly their results as follows.

Assumption 1

The LIBOR Rate Dynamics under the Measure $\mathcal{Q}$

The dynamics of the LIBOR rate $F(t,T)$ under the spot martingale measure $\mathcal{Q}$ is given as follows:
$$dF(t,T) = F(t,T)\gamma(t,T) \cdot \sigma(t,T + \delta)dt + F(t,T)\gamma(t,T) \cdot dZ(t), \tag{2}$$
where $0 \leq t \leq T \leq \mathcal{T}$, $\gamma(t,T): \mathcal{R}_+^2 \rightarrow \mathcal{R}^m$ is a bounded, piecewise continuous, deterministic vector function, and $\sigma(t,\cdot)$ is defined as follows:
$$\sigma(t,T) = \begin{cases} \sum_{j=1}^{\lfloor \delta^{-1}(T-t) \rfloor} \frac{\delta F(t,T-j\delta)}{1+\delta F(t,T-j\delta)} \gamma(t,T-j\delta) & t \in [0, T - \delta] \\ & \& \ T - \delta > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

where $\lfloor \delta^{-1}(T - t) \rfloor$ denotes the greatest integer that is less than $\delta^{-1}(T - t)$.

According to the definition of the bond volatility process (3), $\{\sigma(t,T + \delta)\}_{t \in [0,T+\delta]}$ is stochastic

rather than deterministic. Thus, the stochastic differential equation (2) can not be solved, and thus the distribution of $F(T,T)$ is unknown. However, given a fixed initial time, assumed time 0, we can approximate $\sigma(t,T)$ by $\bar{\sigma}^0(t,T)$ which is defined by

$$\bar{\sigma}^0(t,T) = \begin{cases} \sum_{j=1}^{\lfloor \delta^{-1}(T-t) \rfloor} \frac{\delta F(0,T-j\delta)}{1+\delta F(0,T-j\delta)} \gamma(t,T-j\delta), & t \in [0, T-\delta] \\ & \& \ T-\delta > 0, \\ 0 & otherwise, \end{cases} \tag{4}$$

where $0 \leq t \leq T$. It means that the calendar time of the process $\{F(t,T)\}_{t\in[0,T]}$ in (4) is frozen at its initial time 0 and thus the process $\{\bar{\sigma}^0(t,T)\}_{t\in[0,T]}$ becomes deterministic. By substituting $\bar{\sigma}^0(t,T+\delta)$ for $\sigma(t,T+\delta)$ in the drift term of (2), the drift and the volatility terms in (2) will be deterministic, so we can solve (2) and find the approximate distribution of $F(T,T)$ to be lognormally distributed. This argument is the Wiener chaos order 0 approximation which is first used by BGM (1997) for pricing interest rate swaptions. It was developed further in Brace, Dun, and Barton (1998) and formalized by Brace and Womersley (2000). The accuracy of this approximation for the pricing formula of AIROs (to be derived later) will be shown to be sufficiently accurate.

Proposition 1
The Approximate LIBOR Rate Dynamics under the Measure Q
  The approximate dynamics of the LIBOR rate $F(t,T)$ under the spot martingale measure Q is given as follows:

$$\frac{dF(t,T)}{F(t,T)} = \gamma(t,T) \cdot \bar{\sigma}^0(t,T+\delta)dt + \gamma(t,T) \cdot dZ(t), \tag{5}$$

where $0 \leq t \leq T \leq \mathcal{T}$.

  The following proposition specifies the general rule under which the LIBOR rate dynamics change when the underlying probability measure is altered. This rule is useful for deriving the pricing formulas of the AIROs.

Proposition 2
The Approximate LIBOR Rate Dynamics under Different Measures
  The dynamics of the LIBOR rate $F(t,T)$ under an arbitrary forward martingale measure Q^U is given as follows:

$$\frac{dF(t,T)}{F(t,T)} = \gamma(t,T) \cdot (\bar{\sigma}^0(t,T+\delta) - \bar{\sigma}^0(t,U))dt + \gamma(t,T) \cdot dZ(t), \tag{6}$$

where $0 \leq t \leq \min(U,T) \leq \mathcal{T}$.[4]

  Having briefly introduced the BGM model, we next employ it to derive the pricing formulas of the AIROs in the following section.

3. Pricing Average Interest Rate Options

The payoff structure of an AIRO is defined as follows. Suppose that an AIRO is issued at time 0 $(= t_0)$ and expires at time T $(= t_{n+1})$. The underlying average rate is observed on n different occasions during the life of the option. The observation times are denoted by $\{t_1, t_2, \ldots, t_n\}$ where $t_i < t_{i+1}$ for $i = 0,1,\ldots,n$. For simplicity, we assume $\delta = t_{i+1} - t_i$ for $i = 0,1,2,\ldots,n$. Let K denote the exercise rate. Then, the final payoff of an average interest rate call (AIRC) and an average interest rate put (AIRP) at

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[4] We employ $Z(t)$ to denote an independent m-dimensional standard Brownian motion under an arbitrary measure without causing any confusion.

time T is defined, respectively, as follows:

$$AIRC(T) = \text{Max}(A(T) - K, 0), \tag{7}$$

and

$$AIRP(T) = \text{Max}(K - A(T), 0), \tag{8}$$

where

$$A(T) = \frac{1}{n} \sum_{i=1}^{n} F(t_i, t_i). \tag{9}$$

The final payoffs in (7) and (8) depend on the discrete average of the market-observable LIBOR rates rather than the continuous average of the abstract short rates given in Longstaff (1995). Therefore, our payoff setting is consistent with practical implementation. Moreover, the LIBOR rate in (7) and (8) are specified directly in the LMM rather than transformed from the abstract short rate, which avoids a complicated transformation calculation.

Based on the martingale pricing method, the issuing price of the AIRC and AIRP can be obtained, respectively, by solving the following expectation:

$$B(0, T) E^{Q^T}[\text{Max}(A(T) - K, 0)], \tag{10}$$

and

$$B(0, T) E^{Q^T}[\text{Max}(K - A(T), 0)]. \tag{11}$$

As the problem in pricing ordinary Asian options that the arithmetic average of lognormally-distributed variables is not lognormally distributed, the distribution of $A(T)$ is unknown. Hence, (10) and (11) cannot be solved analytically as closed-form solutions. Levy (1992) employed the Wilkinson approximation method to price analytically Asian currency options. However, his approximation only matches the first two moments and thus leading to some pricing errors in some special financial environment. To increase the pricing accuracy, we adopt the Jarrow and Rudd (1982) approximation method, which uses the first four moments, to deriving the approximate pricing formulas of AIROs, whose accuracy is examined with the Monte Carlo simulation in the next section.

Based on the Wilkinson approximation, we replace the unknown distribution of the arithmetic average of lognormal random variables with a lognormal distribution that has the correct first two moments. In this way, $A(T)$ has an approximately lognormal distribution, and (10) and (11) can be solved.

Hence, we assume that $\ln A(T)$ has a normal distribution with mean M and variance V^2. The moment generating function for $\ln A(T)$ is given by

$$M_{\ln A(T)}(h) = E^{Q^T}[A(T)^h] = \exp(Mh + \frac{1}{2}V^2 h^2). \tag{12}$$

Taking $h = 1$ and $h = 2$ in (12), we obtain two conditions to solve for M and V^2 and the result is given as follows:

$$M = 2\ln E^{Q^T}[A(T)] - \frac{1}{2}\ln E^{Q^T}[A(T)^2], \tag{13}$$

$$V^2 = \ln E^{Q^T}[A(T)^2] - 2\ln E^{Q^T}[A(T)], \tag{14}$$

where $E^{Q^T}[A(T)]$ and $E^{Q^T}[A(T)^2]$ are computed in Appendix A.

With the aforementioned knowledge, (10) can be solved and the pricing formula of an AIRC is given in the following theorem. The proof is provided in Appendix A.

Theorem 1

The Pricing Formula of an Average Interest Rate Call Option

The price of an average interest rate call option at its initial time 0 is given as follows:

$$B(0,T)\left(e^{M+\frac{1}{2}V^2}N(\frac{M-\ln K+V^2}{V}) - KN(\frac{M-\ln K}{V})\right),\qquad(15)$$

where M and V^2 are defined in (13) and (14), and

$$E^{Q^T}[A(T)] = \frac{1}{n}\sum_{i=1}^{n}F(0,t_i)\exp\left(\int_0^{t_i}\gamma(u,t_i)\cdot\mu_0(u;t_{i+1},t_{n+1})du\right),$$

$$E^{Q^T}[A(T)^2] = \frac{1}{n^2}\left(\sum_{i=1}^{n}E^{Q^T}[F(t_i,t_i)^2] + 2\sum_{i=2}^{n}\sum_{j=i}^{n}E^{Q^T}[F(t_{i-1},t_{i-1})F(t_j,t_j)]\right),$$

$$E^{Q^T}[F(t_i,t_i)^2] = F(0,t_i)^2\exp\left(\int_0^{t_i}(2\gamma(u,t_i)\cdot\mu_0(u;t_{i+1},t_{n+1})+\|\gamma(u,t_i)\|^2)du\right),$$

$$E^{Q^T}[F(t_{i-1},t_{i-1})F(t_j,t_j)] = F(0,t_{i-1})F(0,t_j)\exp\left(\int_0^{t_{i-1}}\gamma(u,t_{i-1})\cdot\mu_0(u;t_i,t_{n+1})du\right.$$

$$+\int_0^{t_j}\gamma(u,t_j)\cdot\mu_0(u;t_{j+1},t_{n+1})du + \left.\int_0^{t_{i-1}}\gamma(u,t_{i-1})\cdot\gamma(u,t_j)du\right),$$

$$F(t_i,t_i) = F(0,t_i)\exp\left(\int_0^{t_i}(\gamma(u,t_i)\cdot\mu_0(u;t_{i+1},t_{n+1}) - \frac{1}{2}\|\gamma(u,t_i)\|^2)du\right.$$

$$+\left.\int_0^{t_i}\gamma(u,t_i)\cdot dZ(u)\right),$$

$$\mu_0(t;H_1,H_2) = \bar{\sigma}^0(t,H_1) - \bar{\sigma}^0(t,H_2).$$

As to the pricing of an AIRP, we may apply the put-call parity to solve (11) and the resulting pricing formula is given in the following theorem.

Theorem 2

The Pricing Formula of an Average Interest Rate Put Option

The price of an average interest rate put option at its initial time 0 is given as follows:

$$B(0,T)\left(KN(-\frac{M-\ln K}{V}) - e^{M+\frac{1}{2}V^2}N(-\frac{M-\ln K+V^2}{V})\right).\qquad(16)$$

The pivotal advantage of our pricing model over Longstaff (1995) and CV (1999) is its ease and flexibility in the parameter calibration. Our model can capture the effects of the actual term structures of interest rates and volatilities and the correlations between rates of different terms, which leads to the more accurate pricing formulas of AIROs. In addition, the pricing formulas (15) and (16) bear resemblance to the Black (1976) pricing model for options on futures in the environment of stochastic interest rates, which makes end-users more familiar to employ it.

4. Numerical Study

This section presents the calibration method and numerical examples of our pricing models in the following subsections.

4.1 Calibration Method

As mentioned above, the LMM has the advantage of ease and flexibility in the parameter calibration. This section provides a mechanism to calibrate simultaneously the actual LIBOR zero curve, cap

volatilities, and the correlations of LIBOR rates with different terms. We assume that there are n forward LIBOR rates in an m-factor framework. The calibration procedure is presented in the following steps.

First, we assume that the total volatility structure of $F(t,\cdot)$ is piecewise-constant and depending solely on the time-to-maturity. Table 1 specifies instantaneous total volatilities applied to each period for each rate, which depend on the time-to-maturity of a forward. The volatilities can be calculated from market data and the detailed computational process is presented in Hull (2003).

Table 1: Instantaneous Volatilities of $F(t,\cdot)$

Instant. Total Vol.	Time t $\in (t_0, t_1]$	$(t_1, t_2]$	$(t_2, t_3]$	$\cdots$	$(t_{n-1}, t_n]$
Fwd Rate: $F(t, t_1)$	v_1	Dead	Dead	$\cdots$	Dead
$F(t, t_2)$	v_2	v_1	Dead	$\cdots$	Dead
$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$F(t, t_n)$	v_n	v_{n-1}	v_{n-2}	$\cdots$	v_1

Next, we use the historical price data of the forward LIBOR rates to derive a full-rank $n \times n$ instantaneous-correlation matrix Σ. Thus, Σ is a positive-definite and symmetric matrix and can be written as

$$\Sigma = \Theta \Lambda \Theta',$$

where Θ is a real orthogonal matrix and Λ is a diagonal matrix. Let $H \equiv \Theta \Lambda^{1/2}$ and thus, $HH' = \Sigma$, so that we can find a suitable $n \times m$ matrix G with m-rank ($m \leq n$), such that $\Sigma^G = GG'$ is a m-rank correlation matrix and can be used to mimic the market correlation matrix Σ.

The advantage of this procedure is that we may replace the n-dimensional original Brownian motion $dZ(t)$ with $GdW(t)$ where $dW(t)$ is an m-dimensional Brownian motion. In other words, we change the market correlation structure

$$dZ(t)dZ(t)' = \Sigma dt$$

to a modeled correlation structure

$$GdW(t)(GdW(t))' = GdW(t)dW(t)'G' = GG'dt = \Sigma^G dt.$$

The remaining problem is how to choose a suitable matrix G. Rebonato (1999) proposed the following form for i-th row of G:

$$g_{i,k} = \begin{cases} \cos\theta_{i,k} \ \Pi_{j=1}^{k-1} \ \sin\theta_{i,j} & if \ \ k = 1,2,\ldots,m-1, \\ \Pi_{j=1}^{k-1} \ \sin\theta_{i,j} & if \ \ k = m, \end{cases}$$

for $i = 1,2,\ldots,n$. By finding a $\hat{\theta}$ that solves the following optimization problem

$$\min_{\theta} \ \Sigma_{i,j=1}^{n} \ |\Sigma_{i,j}^B - \Sigma_{i,j}|^2,$$

and substituting $\hat{\theta}$ into G, we obtain a suitable matrix $\hat{G}$ such that $\Sigma^G (= \hat{G}\hat{G}')$ is an approximate correlation matrix for Σ.

Thirdly, $\hat{G}$ can be used to distribute the instantaneous total volatility to each Brownian motion without changing the amount of the instantaneous total volatility.

4.2 Numerical Examination

In this subsection, we provide eight numerical examinations of a 5-year AIROs and a 10-year AIROs based on the market data. The actual market data is from the U.S. Department of the treasury.[5] All the

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[5] The daily treasury yield curve rate shows on the https://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=yieldAll.

data are shown in Table 6 in Appendix C. To construct the initial forward LIBOR rates, we first use the linear interpolation method to find the yield rate of each tenor in 6 months, 1 year, 1.5 years, ..., 9.5 years, 10 years. Then, we transform these yield rates into the initial forward LIBOR rates by

$$F(t, T, T + \delta) = \frac{1}{\delta}\left(\frac{B(t,T)}{B(t,T+\delta)} - 1\right),$$

where $F(t, T, T + \delta)$ is the forward LIBOR rate contracted at time t and applied to the period $[T, T + \delta]$ with $0 \leq t \leq T \leq T + \delta \leq \mathcal{T}$ and

$$B(t, T) = exp(-Y(t,T)T),$$

where $Y(t, T)$ is the yield rate observed at time t and apply at time T. Equip the calibration methods we provide in the previous subsection with the market data, all the parameters in the LMM model can be calibrated as the practical implementation.

With the calibrated parameters, this article employs the Monte Carlo simulation as a benchmark to examine the accuracy of the pricing formula. The simulation is based on 10,000 sample paths. The 5-year and 10-year AIROs with notional principal 1 are priced at different quarterly dates. The strike rate with $K = 0, 50$, and 100 basis points are roughly in-the-money, at-the-money, and out-of-the-money, respectively.

All results are listed in Table 2, Table 3, Table 4, and Table 5. Levy means the price results of our pricing formula with the Levy approach. MC means the price results based on the Monte Carlo simulation method. Furthermore, we also provide the standard error (s.e.) of the Monte Carlo simulation method. All the results in both tables show that whatever in the different moneyness setting, the approximate pricing formula is sufficiently accurate.

Table 2: The 5-yr AIRO

K	LEVY	MC	s.e.
2019/03/01			
	2.2708×10^{-2}	2.2699×10^{-2}	5.0668×10^{-6}
	1.8309×10^{-2}	1.8307×10^{-2}	5.1311×10^{-6}
	1.3909×10^{-2}	1.3902×10^{-2}	5.3089×10^{-6}
2019/06/03			
	1.6286×10^{-2}	1.6288×10^{-2}	3.9054×10^{-6}
	1.1723×10^{-2}	3.1726×10^{-2}	3.9728×10^{-6}
	7.1601×10^{-3}	7.1661×10^{-3}	4.0446×10^{-6}
2019/09/03			
	1.2107×10^{-2}	1.2108×10^{-2}	5.1212×10^{-6}
	7.4335×10^{-3}	7.4320×10^{-3}	5.2713×10^{-6}
	2.7599×10^{-3}	2.7603×10^{-3}	5.3116×10^{-6}
2019/12/02			
	1.5286×10^{-2}	1.5291×10^{-2}	6.6602×10^{-6}
	1.0682×10^{-2}	1.0682×10^{-2}	6.7196×10^{-6}

K	LEVY	MC	s.e.
	6.0781×10^{-3}	6.0667×10^{-3}	6.9344×10^{-6}

[*] The prices of the 5-year AIROs are presented in this table. They are priced via the Levy approach (LEVY) and Monte Carlo simulation (MC) at different quarterly dates over the past 1 year. SE stands for the standard error of MC. The market data used are listed in Table 4 in Appendix C. The notional principal amount is assumed to be $1. The simulation is based on 10000 paths. The strike rates with $K = 0, 50$, and 100 basis points are roughly in-the-money, at-the-money, and out-of-the-money, respectively.

Table 3: The 5-yr AIRO

K	LEVY	MC	s.e.
2020/03/02			
	8.3646×10^{-3}	8.3640×10^{-3}	4.4922×10^{-6}
	3.5798×10^{-3}	3.5761×10^{-3}	4.5839×10^{-6}
	1.2861×10^{-6}	1.3192×10^{-6}	1.9962×10^{-7}
2020/06/01			
	3.1956×10^{-3}	3.1887×10^{-3}	1.1787×10^{-5}
	8.5107×10^{-5}	8.2404×10^{-5}	4.0409×10^{-6}
	6.2689×10^{-7}	6.1962×10^{-7}	4.9115×10^{-7}
2020/09/01			
	2.7106×10^{-3}	2.7089×10^{-3}	7.0589×10^{-6}
	3.8065×10^{-6}	5.1818×10^{-6}	6.6494×10^{-7}
	1.2319×10^{-10}	1.3570×10^{-10}	5.7239×10^{-10}
2020/12/01			
	4.4662×10^{-3}	4.4503×10^{-3}	8.7220×10^{-6}
	1.9132×10^{-4}	1.9364×10^{-4}	4.4547×10^{-6}
	9.8309×10^{-9}	9.3220×10^{-9}	6.5884×10^{-10}

[*] The prices of the 5-year AIROs are presented in this table. They are priced via the Levy approach (LEVY) and Monte Carlo simulation (MC) at different quarterly dates over the past 1 year. SE stands for the standard error of MC. The market data used are listed in Table 4 in Appendix C. The notional principal amount is assumed to be $1. The simulation is based on 10000 paths. The strike rates with $K = 0, 50$, and 100 basis points are roughly in-the-money, at-the-money, and out-of-the-money, respectively.

Table 4: The 10-yr AIRO

K	LEVY	MC	s.e.
2019/03/01			
	2.1183×10^{-2}	2.1182×10^{-2}	5.5248×10^{-6}
	1.7389×10^{-2}	1.7397×10^{-2}	5.8165×10^{-6}

K	LEVY	MC	s.e.
2019/06/03	1.3595×10^{-2}	1.3597×10^{-2}	6.0983×10^{-6}
	1.6813×10^{-2}	1.6806×10^{-2}	4.8942×10^{-6}
	1.2747×10^{-2}	1.2755×10^{-2}	5.1275×10^{-6}
2019/09/03	8.6828×10^{-3}	8.6910×10^{-3}	5.4564×10^{-6}
	1.2547×10^{-2}	1.2554×10^{-2}	6.4411×10^{-6}
	8.2309×10^{-3}	8.2331×10^{-3}	6.7208×10^{-6}
2019/12/02	3.9144×10^{-3}	3.9300×10^{-3}	7.1722×10^{-6}
	1.5397×10^{-2}	1.5405×10^{-2}	8.4191×10^{-6}
	1.1234×10^{-2}	1.1221×10^{-2}	8.8889×10^{-6}
	7.0702×10^{-3}	7.0757×10^{-3}	9.4194×10^{-6}

[*] The prices of the 5-year AIROs are presented in this table. They are priced via the Levy approach (LEVY) and Monte Carlo simulation (MC) at different quarterly dates over the past 1 year. SE stands for the standard error of MC. The market data used are listed in Table 4 in Appendix C. The notional principal amount is assumed to be \$1. The simulation is based on 10000 paths. The strike rates with $K = 0, 50$, and 100 basis points are roughly in-the-money, at-the-money, and out-of-the-money, respectively.

Table 5: The 10-yr AIRO

K	LEVY	MC	s.e.
2020/03/02			
	9.9508×10^{-4}	9.9580×10^{-4}	6.8240×10^{-6}
	5.4717×10^{-4}	5.4852×10^{-4}	7.1030×10^{-6}
	1.0202×10^{-4}	1.0139×10^{-4}	7.0215×10^{-6}
2020/06/01			
	6.4046×10^{-3}	6.4005×10^{-3}	2.3186×10^{-5}
	1.9642×10^{-3}	1.9250×10^{-3}	2.1924×10^{-5}
	2.4727×10^{-4}	2.4572×10^{-4}	9.6510×10^{-6}
2020/09/01			
	6.6260×10^{-3}	6.6214×10^{-3}	1.6874×10^{-5}
	2.0238×10^{-3}	2.0219×10^{-3}	1.6751×10^{-5}
	1.0243×10^{-4}	1.0538×10^{-4}	4.9152×10^{-6}

K	LEVY	MC	s.e.
2020/12/01			
	8.7965×10^{-3}	8.8048×10^{-3}	1.7172×10^{-5}
	4.2364×10^{-3}	4.1889×10^{-3}	1.7957×10^{-5}
	6.0842×10^{-4}	6.0693×10^{-4}	1.1082×10^{-5}

[*] The prices of the 5-year AIROs are presented in this table. They are priced via the Levy approach (LEVY) and Monte Carlo simulation (MC) at different quarterly dates over the past 1 year. SE stands for the standard error of MC. The market data used are listed in Table 4 in Appendix C. The notional principal amount is assumed to be \$1. The simulation is based on 10000 paths. The strike rates with $K = 0, 50$, and 100 basis points are roughly in-the-money, at-the-money, and out-of-the-money, respectively.

5. Conclusion

We have contributed to the literature by providing the pricing formulas for the options on the average interest rate under the LMM. As compared with the earlier interest rate models (Vasicek and HW), the LMM is consistent with the observable yield curve and easier for calibrating the model parameters from market data. In addition, since the underlying interest rates in the LMM are directly forward LIBOR rates, it avoids a complex transformation from the unobservable short rates to the associated LIBOR rates. Thus, pricing AIROs under the LMM is more feasible and tractable for market practitioners.

By comparing with the values obtained via the Monte Carlo, the results of our pricing formulas have been shown to be sufficiently accurate and robust. Hence, the model developed here is suitable for practical implementation.

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Appendix A: Hedging by an AIRC is Cheaper than by a Corresponding Caps

Appendix A will show that hedging periodic interest rate risks with an AIRC is cheaper than with the corresponding interest rate cap. Consider the time flow as follows: $0 < t_0 < t_1 < \ldots < t_n < t_{n+1} = T$, with $\delta = t_i - t_{i-1}$ for $i = 1,2,\ldots,n+1$. Assume that, at the end of each time period $[t_i, t_{i+1}]$ for $i = 1,2,\ldots,n$, we need to pay an interest linked to $L(t_i, t_i)$ based on a principal a_i. We want to hedge the interest rate risk by controlling the average interest rate lower than K. There are two hedging approaches.

Firstly, we hedge the separate interest rate risk at time t_i, for $i = 1,2,\ldots,n$, by using the corresponding caplet with the nominal amount a_i, which is defined as follows:

$$\text{Cap} = \sum_{i=1}^{n} a_i \text{Max}(L(t_i, t_i) - K, 0),$$

which can make sure that each interest rate lower than K, and thus leads the average interest rate lower than K.

Secondly, we hedge the average interest rate risk by employing an AIRC with the nominal amount $\sum_{j=1}^{n} a_j$, which is defined as follows:

$$\text{AIRC} = \begin{cases} \left(\sum_{j=1}^{n} a_j\right)(A(T) - K) & A(T) \geq K, \\ 0 & \text{Otherwise,} \end{cases}$$

where K is an exercise rate and

$$A(T) = \sum_{i=1}^{n} \frac{a_i}{\sum_{j=1}^{n} a_j} L(t_i, t_i).$$

By applying Jensen's inequality, we can show that the AIRC price is cheaper than the price of the corresponding cap, and the proof is given as follows:

$$\text{AIRC} = P(0,T)\left(\sum_{j=1}^{n} a_j\right) \text{E}^{Q^T}\left[\left\{\sum_{i=1}^{n} \frac{a_i}{\sum_{j=1}^{n} a_j} L(t_i, t_i) - K\right\}^+ |\mathcal{F}_0\right]$$

$$= P(0,T)\left(\sum_{j=1}^{n-1} a_j\right)\text{E}^{Q^T}\left[\left\{\left(\sum_{i=1}^{n-1} \left(\frac{a_i}{\sum_{j=1}^{n-1} a_j}\right) L(t_i, t_i) - K\right) + \frac{a_n}{\sum_{j=1}^{n-1} a_j}(L(t_n, t_n) - K)\right\}^+ |\mathcal{F}_0\right]$$

$$\leq P(0,T)\left(\sum_{j=1}^{n-1} a_j\right)\text{E}^{Q^T}\left[\left\{\sum_{i=1}^{n-1} \left(\frac{a_i}{\sum_{j=1}^{n-1} a_j}\right) L(t_i, t_i) - K\right\}^+ |\mathcal{F}_0\right] \quad \text{(By Jensen's Inequality)}$$

$$+ P(0,T)a_n \text{E}^{Q^T}[\{L(t_n, t_n) - K\}^+ |\mathcal{F}_0]$$

$$\ldots\ldots$$

$$\leq \sum_{i=1}^{n} a_i P(0,T) \text{E}^{Q^T}[\{L(t_i, t_i) - K\}^+]$$

$$= \sum_{i=1}^{n} a_i P(0,t_{i+1}) \text{E}^{Q^T}\left[\frac{P(t_{i+1}, t_{i+1})/P(0,t_{i+1})}{P(t_{i+1}, T)/P(0,T)} P(t_{i+1}, T)\{L(t_i, t_i) - K\}^+\right]$$

$$\leq \sum_{i=1}^{n} a_i P(0,t_{i+1}) \text{E}^{Q^{t_{i+1}}}[P(t_{i+1}, T)\{L(t_i, t_i) - K\}^+]$$

$$\leq \sum_{i=1}^{n} a_i P(0,t_{i+1}) \text{E}^{Q^{t_{i+1}}}[\{L(t_i, t_i) - K\}^+], \quad P(t_{i+1}, T) \leq 1$$

$$= \text{Cap},$$

which shows that hedging interest rate risk with an AIRC is cheaper than with the corresponding cap.

Appendix B: The Proof of Theorem 1

Appendix A computes (10) based on the assumption mentioned in Section 3 that

$$ln\, A(T) \sim N(M, V^2),$$

where M and V^2 are defined in (13) and (14). Before the derivation, we present a lemma without proof which is useful in the derivation of (10).[6]

Lemma 1 *If $Y \sim N(0, v^2)$, then the expectation of $E[(Xe^Y - K)^+]$ is given as follows:*

$$Xe^{\frac{1}{2}v^2} N\left(\frac{\ln(\frac{X}{K})+v^2}{v}\right) - KN\left(\frac{\ln(\frac{X}{K})}{v}\right),$$

where X and K are constants.

By Lemma 1, (10) can be derived as follows:

$$B(0,T)\left(e^{M+\frac{1}{2}V^2}N(\frac{M-\ln K+V^2}{V}) - KN(\frac{M-\ln K}{V})\right).$$

The remaining tasks are computing $E^{Q^T}[A(T)]$ and $E^{Q^T}[A(T)^2]$.

According to Proposition 2, under the forward measure $Q^T (= Q^{t_{n+1}})$, the dynamics of $F(t, t_i)$, $i = 1, 2, \ldots, n$, is given as follows:

$$\frac{dF(t,t_i)}{F(t,t_i)} = -\gamma(t,t_i) \cdot (\bar{\sigma}^0(t,t_{n+1}) - \bar{\sigma}^0(t,t_{i+1}))dt + \gamma(t,t_i) \cdot dZ(t),$$

$$= \gamma(t,t_i) \cdot \mu_0(t; t_{i+1}, t_{n+1})dt + \gamma(t,t_i) \cdot dZ(t),$$

where

$$\mu_0(t; t_{i+1}, t_{n+1}) = -(\bar{\sigma}^0(t,t_{n+1}) - \bar{\sigma}^0(t,t_{i+1})).$$

We first derive $E^{Q^T}[A(T)]$ and then $E^{Q^T}[A(T)^2]$.

$$E^{Q^T}[A(T)] = \frac{1}{n}\sum_{i=1}^n E^{Q^T}[F(t_i,t_i)]$$
$$= \frac{1}{n}\sum_{i=1}^n F(0,t_i)\exp\left(\int_0^{t_i} \gamma(u,t_i) \cdot \mu_0(u; t_{i+1}, t_{n+1})du\right).$$
$$E^{Q^T}[A(T)^2] = \frac{1}{n^2}E^{Q^T}[\sum_{i=1}^n F(t_i,t_i)]^2$$
$$= \frac{1}{n^2}\left(\sum_{i=1}^n E^{Q^T}[F(t_i,t_i)^2] + 2\sum_{i=2}^n \sum_{j=i}^n E^{Q^T}[F(t_{i-1},t_{i-1})F(t_j,t_j)]\right),$$

where

$$E^{Q^T}[F(t_i,t_i)^2] = F(0,t_i)^2\exp\left(\int_0^{t_i} (2\gamma(u,t_i) \cdot \mu_0(u; t_{i+1}, t_{n+1}) + \| \gamma(u,t_i) \|^2)du\right)$$

and

$$E^{Q^T}[F(t_{i-1},t_{i-1})F(t_j,t_j)] =$$
$$F(0,t_{i-1})F(0,t_j)\exp\left(\int_0^{t_{i-1}} \gamma(u,t_{i-1}) \cdot \mu_0(u; t_i, t_{n+1})du\right.$$
$$\left. + \int_0^{t_j} \gamma(u,t_j) \cdot \mu_0(u; t_{j+1}, t_{n+1})du + \int_0^{t_{i-1}} \gamma(u,t_{i-1}) \cdot \gamma(u,t_j)du\right).$$

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[6] The proof of Lemma 1 is available upon request from the authors.

Appendix C: The Market Data

Table 6 are the daily treasury yield curve rate disclosed by the U.S. Department of the treasury which are used for the numerical examinations in Section 4.

Table 6: The Daily Treasury Yield Curve Rate

Pricing Date	6-Month	1-Year	2-Year	3-Year	5-Year	7-Year	10-Year
2019-03-01	2.52	2.55	2.55	2.54	2.56	2.67	2.76
2019-06-03	2.31	2.11	1.82	1.79	1.83	1.95	2.07
2019-09-03	1.88	1.72	1.47	1.38	1.35	1.42	1.47
2019-12-02	1.62	1.60	1.61	1.63	1.65	1.77	1.83
2020-03-02	0.95	0.89	0.84	0.85	0.88	1.01	1.10
2020-06-01	0.18	0.17	0.14	0.20	0.31	0.50	0.66
2020-09-01	0.13	0.12	0.13	0.14	0.26	0.46	0.68
2020-12-01	0.10	0.12	0.17	0.22	0.42	0.68	0.92

[*] The daily treasury yield curve rate disclosed by the U.S. Department of the treasury. The treasury yield curve rates are expressed in percentage.